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DYNAMIC VEHICLE-TRACK-STRUCTURE INTERACTION ANALYSIS USING LAGRANGE MULTIPLIERS

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ABSTRACT

In the design of bridges and other overhead structures that support high-speed trains, particularly complex or unproven structural systems, an analysis accounting for the dynamic interaction between the train and the structure is required. This is termed dynamic vehicle-track-structure interaction (VTSI). In this paper, we present an approach to such analysis where the bridge structure and the train are modeled independently and coupled by time-varying kinematic constraints. Specifically, displacements of the train wheels are constrained to be equal to the track displacements at their locations. The contact forces between the wheels and the track appear in this formulation as Lagrange multipliers corresponding to these constraints. Particular consideration is needed in the interpolation of the constraints to avoid spurious oscillations in the Lagrange multipliers, and consequently accelerations, which we solve using a cubic B-spline constraint interpolation. For three-dimensional bridge and train models, including curved bridges, we adopt a corotational approach to representing the deformations of the suspension system within the vehicle body. We also use an energy-conserving time integration scheme that reduces to Newark's constant average acceleration method in the absence of constraints. Implementation in the structural analysis software LARSA 4D is discussed, with particular emphasis on implications for users.

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Dynamic Vehicle-Track-Structure Interaction Analysis using Lagrange Multipliers

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ABSTRACT

In the design of bridges and other overhead structures that support high-speed trains, particularly complex or unproven structural systems, an analysis accounting for the dynamic interaction between the train and the structure is required. This is termed dynamic vehicle-track-structure interaction (VTSI). In this paper, we present an approach to such analysis where the bridge structure and the train are modeled independently and coupled by time-varying kinematic constraints. Specifically, displacements of the train wheels are constrained to be equal to the track displacements at their locations. The contact forces between the wheels and the track appear in this formulation as Lagrange multipliers corresponding to these constraints. Particular consideration is needed in the interpolation of the constraints to avoid spurious oscillations in the Lagrange multipliers, and consequently accelerations, which we solve using a cubic B-spline constraint interpolation. For three-dimensional bridge and train models, including curved bridges, we adopt a corotational approach to representing the deformations of the suspension system within the vehicle body. We also use an energy-conserving time integration scheme that reduces to Newark’s constant average acceleration method in the absence of constraints. Implementation in the structural analysis software LARSA 4D is discussed, with particular emphasis on implications for users.

Introduction

When bridges and structures that support elevated tracks are designed to carry high-speed trains, it is often necessary to consider the effect of dynamic interaction between the train, tracks and the structure. Conceptually, this interaction could be thought of in the following manner. As a train traverses a bridge, the vertical deflections of the bridge as well as irregularities in the track act as support displacement input to the train at the wheels. The ensuing dynamics of the train in turn cause time-varying forces and vibration in the bridge. This is termed dynamic vehicle-track-structure interaction (VTSI). It is particularly significant for high-speed trains since the dynamic forces applied on the bridge are more likely to be in the range of resonant frequencies of the bridge. Undue deformations and accelerations resulting from dynamic VTSI could result in excessive changes in track geometry and rail stress, dynamic magnification of loads and passenger discomfort. Therefore, extensive dynamic analysis considering VTSI is recommended to verify track safety and passenger comfort, and ensure certain design limits particularly for complex and non-standard structural systems (example, [1]).

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A number of approaches have been adopted for dynamic VTSI analysis. A description of some of these is presented in the book [2]. On the one hand, simple models are used that represent the effect of interaction by means of dynamic impact factors (example [3]). At the other extreme are sophisticated models intended to obtain highly detailed information on various aspects of the response of train cars (example [4]). Such models are necessary in the mechanical design train of high-speed train cars. Some approaches also employ specialized bridge-train interaction elements (example [5]).

We seek a formulation that is in between these extremes in terms of detail for use in bridge analysis and design. We wish to include sufficient detail so that the train dyanmics that is important in meaningfully computing bridge response can be modeled. We also wish to represent track irregularities. Further, we take an approach characterized by the following features.

1. We describe the dynamics of the train and the bridge independently, and represent coupling between the two using *constraints*. This results in the forces between the bridge and the train taking the form of *Lagrange multipliers*.
2. We use a *predictor-corrector* format in implementing the numerical integration schemes for the bridge and train systems.
3. The effect of the bridge dynamics on the train dynamics is represented by means of an *influence matrix* for the bridge.

We find that these result in greatest modularity and ease of implementation. As will be seen later, the bridge-train contact forces are a direct product of the computation.

Mathematical formulation

A conceptual model of a high-speed train passing over a bridge is shown in Figure 1. As discussed above, the bridge and train are modeled completely independently, and are coupled together by kinematic constraints.

Bridge and train models

The bridge may be represented as is usually done for dynamic structural analysis using beam elements, truss elements, plate elements, cable elements, dashpots or other finite elements as appropriate. Our VTSI formulation imposes no inherent additional restrictions on how the bridge structure may be modeled. If a more detailed model explicitly including the rails, ballast etc. is desired, these could be represented by suitable finite element models as well.

The train is modeled as a sequence of cars, with each car consisting of an assembly of rigid bodies, springs and dashpots. In fact, as discussed later, the train could be considered as another independent structure (with stiff members used to approximate rigid bodies).

Kinematic constraints

The bridge and train models are coupled together using a constraint condition, namely that the displacement of each wheel of the train is equal to the displacement of the bridge at the location of the wheel. Interestingly, track irregularities may be introduced naturally through these constraints as well. When track irregularities are present, the difference between the wheel displacement and the corresponding bridge displacement equals the track irregularity at that location.

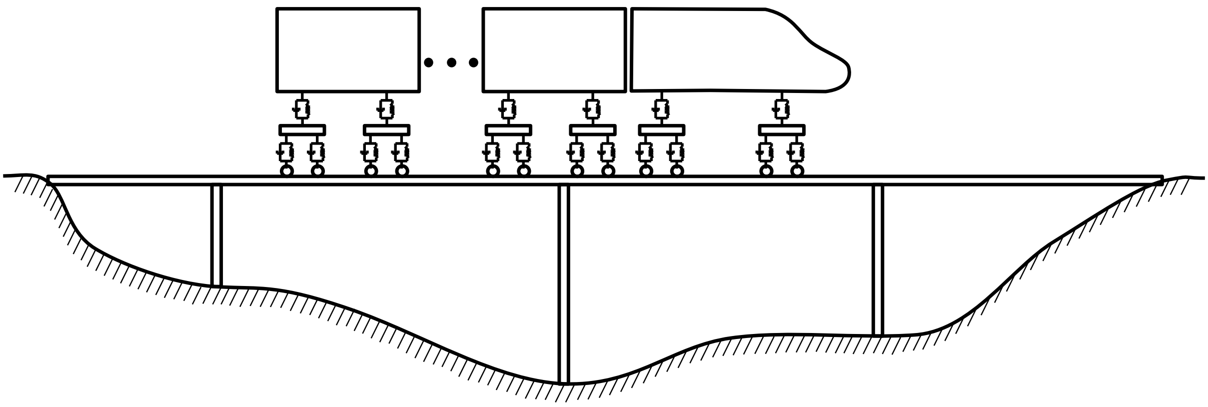

Figure 1. Conceptual model of train traversing a bridge

Additional simplifications

In the initial implementation described in this paper, we use the following additional simplifications for convenience, although these are not necessary for the general formulation.

1. The train moves on a straight-line path along the bridge, so that the train dynamics is entirely planar, and the train loads on the bridge are in the global z direction.
2. The train and bridge displacements are small, so that linearized kinematics can be used for both. Material behavior is also linear. As described later, when curved paths are considered, geometric nonlinearity naturally enters the picture.
3. Train wheels do not lose contact with the bridge. If tension forces develop between the bridge and the train, we deem the analysis invalid. Contact seperation can be readily included in our formulation within the framework of kinematic constraints using inequality constraints.
4. When a wheel is outside the span of the bridge, we take its displacement to be zero.
5. There is no simultaneous uniform base acceleration input to the bridge. As a result, all displacement, of the train and bridge, can be taken as total displacements with respect to an inertial frame.

Dead loads are applied to the bridge and train models prior to dynamic VTSI analysis using an initial static analysis stage.

Governing equations

Using the bridge and train models and kinematic constraints described above, we obtain the following governing equations for dynamic VTSI.

$$\begin{aligned}
&\text{Train equation of motion:} && M^{\mathrm{t}}\ddot{u}^{\mathrm{t}} + C^{\mathrm{t}}\dot{u}^{\mathrm{t}} + K^{\mathrm{t}}u^{\mathrm{t}} + L^{\mathrm{t}\mathrm{T}}\lambda = P^{\mathrm{t}} \\
&\text{Bridge equation of motion:} && M^{\mathrm{b}}\ddot{u}^{\mathrm{b}} + C^{\mathrm{b}}\dot{u}^{\mathrm{b}} + K^{\mathrm{b}}u^{\mathrm{b}} + L^{\mathrm{b}}(t)^{\mathrm{T}}\lambda = P^{\mathrm{b}} \\
&\text{Kinematic constraint:} && L^{\mathrm{t}}u^{\mathrm{t}} + L^{\mathrm{b}}(t)u^{\mathrm{b}} = -\rho(t)
\end{aligned} \tag{1}$$

Here, u^{t} and u^{b} are the train and bridge displacements respectively, M^{t}, C^{t} and K^{t} are the mass, stiffness and damping matrices of the train model, M^{b}, C^{b} and K^{b} are the corresponding matrices of the bridge model, and P^{t} and P^{b} are the external loads on the train and bridge models respectively. In the third equation that represents the kinematic constraint, L^{t} is a matrix that extracts the wheel displacements (with a negative sign) from the vector of displacements of the train model; $L^{\mathrm{b}}(t)_{ij}$ is an *influence matrix*, containing the vertical displacement of the bridge at the location of wheel i at time t, due to unit displacement at degree of freedom j of the bridge. ρ is the vector of track irregularities at the wheel locations. λ is the vector of contact forces between the train wheels and the bridge (positive downward on the bridge/upward on the wheels). It is the *Lagrange multiplier* corresponding to the kinematic constraint. We see that the terms $L^{\mathrm{tT}}\lambda$ (forces applied by bridge on wheels) and $L^{\mathrm{b}}(t)^{\mathrm{T}}\lambda$ (force applied by train on bridge at wheel locations) appear in the equations of motion of the train and bridge respectively.

Special consideration is necessary in the construction of the matrix $L^{\mathrm{b}}(t)$. Roughly speaking, we note that accelerations of the train and bridge are related by the second derivative with respect to time of the third of equations (1), which would contain the second derivative of $L^{\mathrm{b}}(t)$. Thus if the $L^{\mathrm{b}}(t)$ is not twice continuously differentiable, we observe spurious discontinuities in computed accelerations and contact forces λ. This would be the case if $L^{\mathrm{b}}(t)$ were constructed using cubic polynomials that are used for interpolation in beam elements in the bridge model, for with such functions the displacement and rotation are continuous but the curvature (second derivative) is not. Instead we use B-spline functions that are globally twice continuously differentiable to construct $L^{\mathrm{b}}(t)$. In the next section, we present the numerical algorithm used to compute a solution to equation (1).

Solution algorithm

We discretize equation (1) in time for numerical integration using a method proposed by Bauschau [6] for constrained dynamical systems. When there is no damping, this method is energy conserving. Interestingly, the equations of motion (the first two of equations (1)) are enforced at the half time-step, and the kinematic constraint (the last of equations (1)) at the end of the time step. The task is, given the bridge and train displacements and velocities at time *n*, to compute the same quantities at time *n*+1 and the contact forces at time *n*+1/2. The kinematics are discretize as follows (similar to Newmark's method).

$$
\begin{aligned}
u^{\mathrm{t}}_{n+1} &= u^{\mathrm{t}}_{n} + \Delta t \frac{v^{\mathrm{t}}_{n+1} + v^{\mathrm{t}}_{n}}{2} \\
u^{\mathrm{b}}_{n+1} &= u^{\mathrm{b}}_{n} + \Delta t \frac{v^{\mathrm{b}}_{n+1} + v^{\mathrm{b}}_{n}}{2}
\end{aligned} \tag{2}
$$

where v^{t} and v^{b} are the train and bridge veocities respectively, and subscripts denote increment index. Equations (1) are discretized as

$$
\begin{aligned}
& M^{\mathrm{t}} \frac{v^{\mathrm{t}}_{n+1} - v^{\mathrm{t}}_{n}}{\Delta t} + C^{\mathrm{t}} \frac{v^{\mathrm{t}}_{n+1} + v^{\mathrm{t}}_{n}}{2} + K^{\mathrm{t}} \frac{u^{\mathrm{t}}_{n+1} + u^{\mathrm{t}}_{n}}{2} + L^{\mathrm{tT}} \lambda_{n+1/2} = \frac{P^{\mathrm{t}}_{n+1} + P^{\mathrm{t}}_{n}}{2} \\
& M^{\mathrm{b}} \frac{v^{\mathrm{b}}_{n+1} - v^{\mathrm{b}}_{n}}{\Delta t} + C^{\mathrm{b}} \frac{v^{\mathrm{b}}_{n+1} + v^{\mathrm{b}}_{n}}{2} + K^{\mathrm{b}} \frac{u^{\mathrm{b}}_{n+1} + u^{\mathrm{b}}_{n}}{2} + L^{\mathrm{b}}(t_{n+12})^{\mathrm{T}} \lambda_{n+1/2} = \frac{P^{\mathrm{b}}_{n+1} + P^{\mathrm{b}}_{n}}{2} \\
& L^{\mathrm{t}} u^{\mathrm{t}}_{n+1} + L^{\mathrm{b}}(t_{n+1}) u^{\mathrm{b}}_{n+1} = -\rho(t_{n+1})
\end{aligned} \tag{3}
$$

This reduces to Newmark's method with parameters 0.25, 0.5 in the absence of constraints. We now define

$$
\begin{aligned}
\bar{v}^{\mathrm{t}} &= \frac{v^{\mathrm{t}}{}_{n+1} + v^{\mathrm{t}}{}_{n}}{2} \\
\bar{v}^{\mathrm{b}} &= \frac{v^{\mathrm{b}}{}_{n+1} + v^{\mathrm{b}}{}_{n}}{2}
\end{aligned} \tag{4}
$$

It is most convenient to rearrange equation **Error! Reference source not found.** so as to solve for $(\bar{v}^{\mathrm{t}}, \bar{v}^{\mathrm{b}}, \lambda_{n+12})$. Doing so results in the following linear system of equations.

$$
\begin{aligned}
&\bar{M}^{\mathrm{t}}\bar{v}^{\mathrm{t}} + \frac{\Delta t}{2} L^{\mathrm{tT}} \lambda_{n+12} = a^{\mathrm{t}} \\
&\bar{M}^{\mathrm{b}}\bar{v}^{\mathrm{b}} + \frac{\Delta t}{2} L^{\mathrm{b}}(t_{n+12})^{\mathrm{T}} \lambda_{n+12} = a^{\mathrm{b}} \\
&\frac{\Delta t}{2} L^{\mathrm{t}}\bar{v}^{\mathrm{t}} + \frac{\Delta t}{2} L^{\mathrm{b}}(t_{n+1})\bar{v}^{\mathrm{b}} = b
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
\bar{M}^{\mathrm{t}} &= M^{\mathrm{t}} + \frac{\Delta t}{2} C^{\mathrm{t}} + \frac{\Delta t^2}{4} K^{\mathrm{t}} \\
\bar{M}^{\mathrm{b}} &= M^{\mathrm{b}} + \frac{\Delta t}{2} C^{\mathrm{b}} + \frac{\Delta t^2}{4} K^{\mathrm{b}} \\
a^{\mathrm{t}} &= M^{\mathrm{t}} v^{\mathrm{t}}{}_{n} + \frac{\Delta t}{2} \frac{P^{\mathrm{t}}{}_{n+1} + P^{\mathrm{t}}{}_{n}}{2} - K^{\mathrm{t}} u^{\mathrm{t}}{}_{n} \\
a^{\mathrm{b}} &= M^{\mathrm{b}} v^{\mathrm{b}}{}_{n} + \frac{\Delta t}{2} \frac{P^{\mathrm{b}}{}_{n+1} + P^{\mathrm{b}}{}_{n}}{2} - K^{\mathrm{b}} u^{\mathrm{b}}{}_{n} \\
b &= -\frac{1}{2}\left[L^{\mathrm{t}} u^{\mathrm{t}}{}_{n} + L^{\mathrm{b}}(t_{n+1}) u^{\mathrm{b}}{}_{n} + \rho(t_{n+1}) \right]
\end{aligned} \tag{6}
$$

We solve this (at each time increment) in such a manner that the train model is treated completely independent of the structural analysis code where the bridge is modeled. This is accomplised by means of a *predictor-corrector* scheme. The structural analysis code simply calls some functions for the train model-related computations. We next describe the implementation of this algorithm in the structural analysis software LARSA 4D.

Implementation in LARSA 4D

The VTSI procedure described in the last section has been implemented within the commercial software package LARSA 4D as an extension of the linear time history analysis option. In place of time history excitation curves, the user instead provides as input for VTSI:

- the stiffness, mass, and damping of vehicles
- a track
- the velocity of the vehicles
- a vertical track irregularity profile (optional)

A VTSI project will contain both the bridge and one or more vehicles all in the same file. In other words, the vehicles are explicitly represented with joints and structural elements in the same input file as the bridge. The bridge and the vehicles will not be structurally connected in the project file, however. The changing connectivity between the vehicles and track is managed during the analysis. It does not matter where the vehicles are in space in relation to the bridge or what coordinates are given to the vehicles.

While we envision providing the user with a standard database of vehicles in the future, we chose to have the vehicles explicitly represented in the project file to offer the most flexibility both in input and output. In input, the user is able to define any structure for the vehicles that would be permitted in a standard linear time history analysis using any linear elements (including the beam, spring, shell, and linear viscous dashpot). And in output, because the results for both the bridge and vehicle are tied to records in the same project file, the user may use any LARSA 4D post-processing tool in the regular way, such as the tool for creating time series plots normally used in a standard time history analysis.

LARSA 4D identifies the joints and elements as being a part of the vehicles (and not the bridge) through the use of special displacement user coordinate systems assigned to the vehicles' wheels. These coordinate systems serve both to identify the vehicles to the program as well as to anchor the arbitrary coordinates of vehicles to a reference point.

The track is defined as a list of beam elements that the vehicles' path follows. The two-dimensional nature of the VTSI analysis has two consequences for the track. First, the track is positioned at the vehicle centerline. If the bridge model includes rails, the rails must be rigidly connected to dummy elements at the vehicle centerline to form the track. Second, the track must be perfectly straight. But note that beyond these requirements for the track, the remainder of the bridge model may be fully three-dimensional. Additionally, a vertical profile on the track may be created through loading, and a vertical track irregularity curve can be entered to offset the vehicle's path from the members that form the track. The irregularity curve is entered within LARSA 4D's database editor.

To run a VTSI analysis, the user creates a new load case and sets its type to Time History as in a normal time history analysis. However, inside the load case, the user uses the "moving loads" spreadsheet to enter the track and the vehicle velocity, in a similar manner for how the user would prepare a live load analysis. Joint loads and automatic self-weight computation (for both the bridge and the vehicles) are also permitted. Finally the user runs a Linear Time History Analysis from the Analysis menu. The VTSI analysis is triggered by the presence of the "moving load" in the time history load case.

As the vehicles and bridge are explicitly modeled in the structure, results for both the vehicles and bridge are accessed as in any other LARSA 4D analysis. All results typically available in a static analysis, plus those available in a linear time history analysis, will be available. The changing connectivity between the vehicles and the track is given in a special output spreadsheet. The "lambda" term in the VTSI formulation is reported as joint reactions on the wheels.

Example

We have run the VTSI analysis for a number of scenarios including the one described in this section. In this example, a two-car train passes at 200 km/h over a 300 m two-span straight bridge.

The two cars each are 20 m in length, have four wheels, are identical, and are spaced 10.5 m apart. Figure 2 shows the structure of each car. The wheels are connected by springs to bogies, which are in turn suspended from the train car by additional springs. Linear viscous dashpots acting in the vertical direction are coincident with the car-bogie suspension and act to dampen the connections. The bogies and car bodies are rigid. Mass is specified at the wheels and at the middle of the bogies and car bodies.

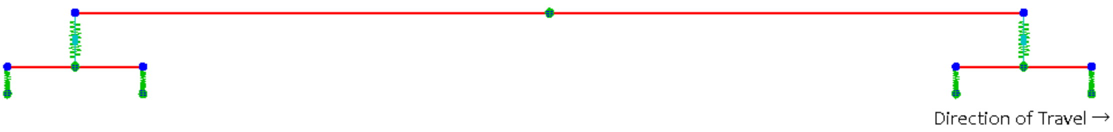

Figure 2. One of the two train cars in the model

The bridge is composed of two continuous spans of 150 meters which also make up the track. To ensure that the train cars enter and exit the track smoothly, the ends of the track are fixed. In addition, a track irregularity curve is applied. This curve specifies a vertical deviation of the track from -1.0 to 1.2 centimeters. The curve is shown in Figure 3.

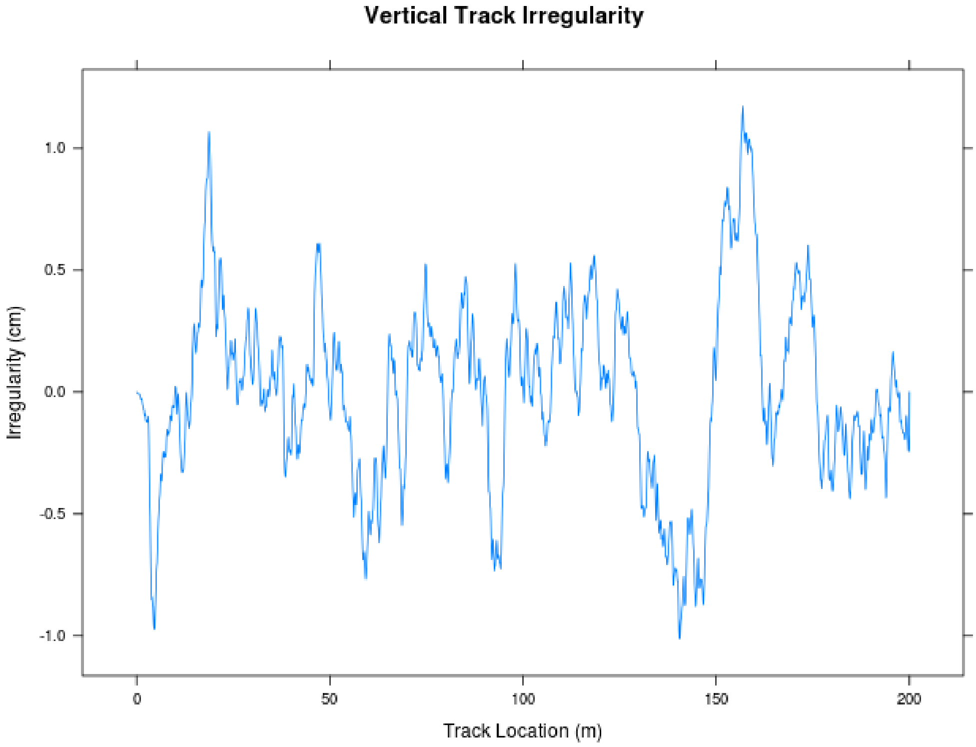

Figure 3. Vertical track irregularity profile

The model additionally includes a uniform load on the track and vertical loading at the locations of mass in the vehicles.

This example problem was analyzed using our implementation of the VTSI algorithm in LARSA 4D using a 0.01 sec time step and an ending time of 6 sec. (The last wheel of the train exits the track at 4.51 sec.) The analysis takes two minutes to run on typical desktop hardware.

We used LARSA 4D's graphing tool to produce the plot in Figure 4 of the vertical displacement at the center of the front car as a function of time.

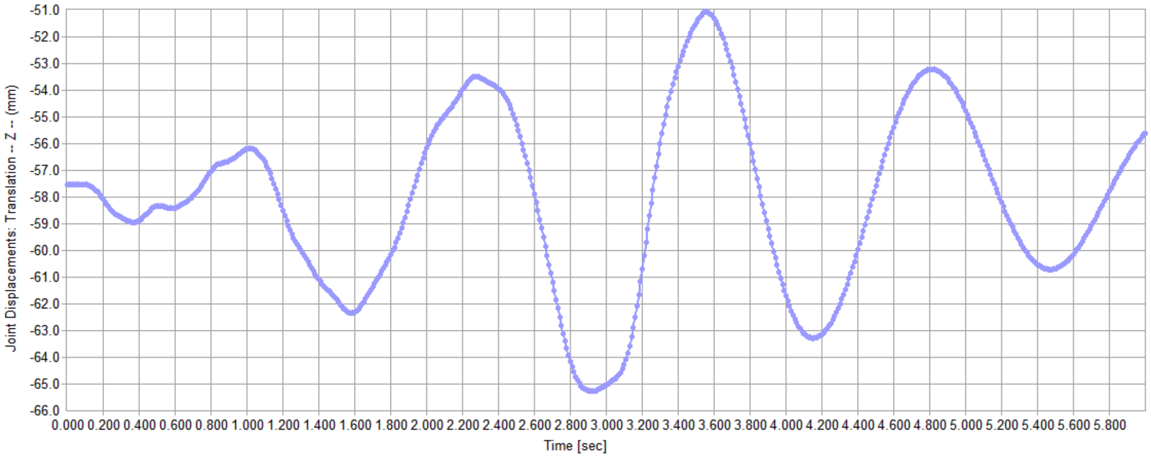

Figure 4. Vertical displacement at the center of the front car as a function of time. Although the cars have left the track at 4.51 sec, the vehicle remains a part of the analysis until the final time step with wheels assumed to have zero displacement.

Extension to three-dimensional models and curved paths

Although the example presented is of a two-dimensional model, the algorithm described above applies as is to three-dimensional models. However paths containing horizontally or vertically curved segments require additional considerations. For example, to properly account for centrifugal and Coriolis forces resulting from the train traversing such curves, large geometry changes in the bridge must be modeled. Therefore, geometric nonlinearity naturally enters the picture. This can however be incorporated by allowing for large rigid-body displacements of the train cars while restricting displacements within each car to be small using a corotational formulation. We are currently implementing such a formulation.

Summary

We have developed a formulation for dynamic bridge-train interaction that can be used for dynamic VTSI analysis of bridges with high-speed trains. The formulation consists of representing the train and bridge models independently and coupling them using kinematic constraints. The contact forces between the train and the bridge at the wheel locations appear as Lagrange multipliers. Track irregularities can also be considered through the kinematic constraints. Efficient solution, where the largest linear system solved is the size of the bridge stiffness matrix, is accomplished through a predictor-corrected scheme. A special energy-conserving time integration algorithm and B-spline approximation for the kinematic constrains are used. The formulation is implemented in the structural analysis software LARSA 4D, which allows for modeling both the train and bridge as independent structures and applying the kinematic constraints. This facilitates dynamic VTSI analysis for bridge engineers. Extensions to three-dimensional models and curved paths are underway.

References

1. California High-Speed Train Project, Technical Memorandum TM 2.10.10. “Track-Structure Interaction”, Parsons Brinckerhoff, December 2013.
2. Yang, Y.-B., Yao, Z., Wu, Y. S. Vehicle-bridge interaction dynamics with applications to high-speed railways, World Scientific, River Edge, NJ, 2004.
3. Frýba, L. A rough assessment of railway bridges for high speed trains. *Engineering Structures*

2001; 23(5): 548-556.
4. Zacher, M., Ambrogi, F., Fassbender, F. Vehicle/Bridge Interaction Model Using DB AG Internal Software. *15th European ADAMS Users' Conference* 2000; Rome, Italy.
5. Guo, W. W., Xia, H. De Roeck, G., Liu, K. Integral model for train-track-bridge interaction on the Sesia viaduct: Dynamic simulation and critical assessment. *Computers & Structures* 2012; 112–113(0): 205-216.
6. Bauchau, O. A. A self-stabilized algorithm for enforcing constraints in multibody systems. *International Journal of Solids and Structures* 2003; 40(13–14): 3253-3271.